this manner, the ratio $\delta_1/\delta_2$ for uniform and nonuniform specimens, with m = 10, we obtain the value 1.055 for complex (3.1). In the same manner, we verify that complex (3.1) is greater than 1 for all $1.43 \leqslant m < \infty$.

Thus, there exists a range of values of m, in which theory agrees qualitatively with experiment, and in this range there exists a value of m that ensures quantitative agreement.

We consider the results obtained here as a significant indication of the existence of surface nonuniformity in solid bodies.

## LITERATURE CITED


1. C. Kittel, Introduction to Solid State Physics, (1976).
2. Tables of Physical Quantities (Handbook) [in Russian], Atomizdat, Moscow (1976).
3. V. F. Babich and Yu. S. Lipatov, "Dynamic method for determining the mechanical properties of a surface polymer layer," Mekh. Polim., No. 3 (1969).
4. Yu. P. Stepanenko, "Dynamometric trapping system for testing specimens for uniaxial tension," in: Plastics in Machine Construction and in Railroad Transport [in Russian], Transport, Moscow (1967).
5. Yu. P. Stepanenko, "Differential method for measuring tension in polymers," Mekh. Polim. No. 3 (1974).
6. G. P. Aleksandrova, "Contact problems in bending of a slab lying on an elastic foundation," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 1 (1973).
7. G. P. Aleksandrova, "Effect of surface tension forces on the contact rigidity," in: Calculation of Shells and Plates [in Russian], Rostov-on-Don, Izd. RISI (1975).
8. I. I. Vorovoch, V. M. Aleksandrov, and V. A. Babeshko, Nonclassical Mixed Problems in the Theory of Elasticity [in Russian], Nauka, Moscow (1974).


# SELF-MODELING PROBLEMS IN THE DYNAMIC BENDING OF BEAMS

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The study of self-similar motions of continuous media is very fruitful [1]. The set of self-similar problems is limited by restrictions on the dimensionalities of the characteristic quantities. In those cases when these requirements are satisfied, the mathematical aspect of the problem can be greatly simplified.

In the present work, we examine the self-similar problems of dynamic bending of beams, satisfying the dynamic Bernoulli–Euler equation. For infinitely long beams, self-similar solutions and solutions including self-similar components are known [2-8]. All these solutions have been obtained, however, without using the properties of self-similarity. In the present work, we propose a general method that permits studying a wide class of self-similar solutions. Known self-similar solutions can be obtained as particular cases based on this method. In addition, methods are established for solving problems of bending of beams with moving supports, whose motion occurs within a regime that retains the self-similarity of the problem. We will refer to this regime as the self-similar regime of motion. The bending of a beam under the action of a force that moves along the beam in the self-similar regime indicated is investigated for the first time.

The properties of self-similarity were used previously in [9] for studying the deformation of membranes with movable boundaries.

1. Let us examine the equation for bending of a beam

$$EI\partial^4 w/\partial x^4 + m\partial^2 w/\partial t^2 = q\ (x, t), \tag{1.1}$$

where w is the deflection; t, time; x, coordinate; E, modulus of elasticity of the material; I, moment of inertia of a section; m, an adjustable mass; q, an adjustable load.

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Let us first consider the homogeneous equation. For the homogeneous equation, the deflection is a function of the following determining parameters: EI, m, t, and x. From these parameters, it is possible to form only one independent dimensionless combination, which simultaneously includes the variables x and t. We write this combination in the form

$$\xi = x / (2\sqrt[4]{a}\,\sqrt{t}),\quad a = EI/m. \tag{1.2}$$

The function obtained indicates the self-similarity of the motion [1]. For the inhomogeneous equation (1.1), the self-similarity of the solution is retained if the load q can be represented in the form

$$q = q_* t^{\beta} f(\xi), \tag{1.3}$$

where $q_*$ is a constant; $f(\xi)$, dimensionless function of the variable $\xi$; and $\beta$, an exponent.

The appearance of a new dimensional quantity $q_*$, independent of the constants EI and m, does not destroy self-similarity, since the problem is linear and $q_*$ enters into the final expression as a factor.

The boundary and initial conditions contain additional dimensional quantities. For example, concentrated forces can act on sections of the beam or velocities can be given that vary as a power law function of time

$$P = P_* t^{\gamma},\quad v = v_* t^{\delta}, \tag{1.4}$$

where $\gamma$ and $\delta$ are exponents; $P_*$ and $v_*$ are constant quantities.

The dimensionalities of the constants $P_*$ and $v_*$ must be independent of the dimensionalities of the previously introduced quantities, in order that the problem remain self-similar. This is equivalent to imposing restrictions on the magnitudes of the exponents $\gamma$ and $\delta$. If in Eq. (1.1) the right side equals zero, then the one-dimensional constant, depending on $q_*$ and entering as a factor in the final solution, which can be used for satisfying an independent boundary or initial condition, is freed. In this case, in satisfying one of conditions (1.4), one of the exponents $\gamma$ or $\delta$ can be chosen arbitrarily. The rest of the conditions must have dependent dimensionalities or be null conditions.

The conditions of the problem cannot contain characteristic dimensions, which occurs for infinite and semiinfinite beams. In addition, it is possible to give special moving boundary conditions, corresponding to self-similar regimes. We note that due to the linearity of Eq. (1.1) many nonself-similar solutions can be obtained by superposition of self-similar solutions.

2. Self-similar solutions are sought in the form

$$w = At^{\alpha}\varphi(\xi), \tag{2.1}$$

where $\alpha$ is an exponent; A, a dimensional constant; $\varphi(\xi)$, a dimensionless function of the variable $\xi$. Expression (2.1) is substituted into Eq. (1.1), in which the right side has the form (1.3). In calculating the derivatives, the variable $\xi$ is viewed as a function of x and t. As a result of the substitution, an ordinary differential equation is obtained:

$$\varphi^{IV} + 4\xi^2\varphi'' + (12 - 16\alpha)\xi\varphi' + 16\alpha(\alpha - 1)\varphi = 16f(\xi), \tag{2.2}$$

in which the derivatives are taken with respect to the variable $\xi$.

Here, it is necessary to set $A = q_* m^{-1}$ and $\alpha = \beta + 2$. If, on the other hand, Eq. (2.2) is homogeneous, then the quantity $\alpha$ and the dimensionality A are determined from an additional condition (e.g., a boundary condition).

For example, let a concentrated force act on the section x

$$P = P_* h(t), \tag{2.3}$$

where h(t) is a Heaviside unit function. In this case, the third derivative of the deflection is discontinuous at the point at which the force is applied:

$$EI\left(\left.\frac{\partial^3 w}{\partial x^3}\right|_{x+0} - \left.\frac{\partial^3 w}{\partial x^3}\right|_{x-0}\right) = P_*.$$

Substituting here (2.1), we obtain

$$0.125A\,(EIm^3)^{1/4}\, t^{\alpha-3/2}\left(\varphi'''(\xi+0) - \varphi'''(\xi-0)\right) = P_*, \tag{2.4}$$

where the variable $\xi$ has the meaning of the corresponding coordinate x of the point of application of the force. The conditions, allowing for the transition from the coordinate x to the specific value of $\xi$ in (2.4), are indicated in what follows. Expression (2.4) must give the condition for $\varphi'''$ which does not explicitly depend on time and which does not have any dimensionality. From this it follows that $\alpha = 1.5$, $[A] = [P_*][EIm^3]^{-1/4}$. Here, the square brackets indicate the dimensionality of the corresponding quantity. The choice of the quantity A, with the exception of the dimensionality, is arbitrary, since in what follows it can be combined with an arbitrary integration constant for Eq. (2.2). In the case being examined, the constant (A) can be chosen so that in (2.4) the discontinuity in the derivative $\varphi'''$ would be equal to unity. From here it follows that

$$A = 8P_* (EIm^3)^{-1/4}. \tag{2.5}$$

In what follows, we will investigate Eq. (2.2) without the right side, omitting the word homogeneous in referring to it.

The solution of (2.2) was sought by substituting an expansion in powers of $\xi$. The series so obtained can be expressed for the most useful values of $\alpha$ in terms of trigonometric functions and Fresnel integrals [10]

$$S(\xi) = \frac{2}{\sqrt{2\pi}} \int_0^\xi \sin y^2 dy, \quad C(\xi) = \frac{2}{\sqrt{2\pi}} \int_0^\xi \cos y^2 dy.$$

The most widely used values of $\alpha$ are $\alpha = 1.5$; 1; 0.5. These values correspond to the action of a concentrated force ($\alpha = 1.5$) on the beam, moment ($\alpha = 1$), and setting the value of the velocity in the section of the beam ($\alpha = 1$), varying in time according to the unit function [2-4]. For pulsed actions [6], solutions of (2.2) may be required with $\alpha = 0.5$.

Let us write the fundamental system of solutions for (2.2) for the values of $\alpha$ indicated:

for $\alpha = 1.5$

$$\begin{gathered}\varphi_1 = -\sqrt{2\pi}\xi^3 C(\xi) + 1.5\sqrt{2\pi}\xi S(\xi) + \xi^2 \sin\xi^2 + \cos\xi^2, \quad \varphi_2 = \xi, \\ \varphi_3 = \frac{1}{6}\left[\sqrt{2\pi}\xi^3 S(\xi) + 1.5\sqrt{2\pi}\,\xi C(\xi) + \xi^2\cos\xi^2 - \sin\xi^2\right], \quad \varphi_4 = \frac{1}{6}\xi^3;\end{gathered} \tag{2.6}$$

for $\alpha = 1$

$$\begin{gathered}\varphi_1 = 1, \quad \varphi_2 = 0.5[\sqrt{2\pi}\xi^2 S(\xi) + 0.5\sqrt{2\pi} C(\xi) + \xi\cos\xi^2], \\ \varphi_3 = 0.5\xi^2, \quad \varphi_4 = 0.25[\sqrt{2\pi}\xi^2 C(\xi) - 0.5\sqrt{2\pi} S(\xi) - \xi\sin\xi^2];\end{gathered} \tag{2.7}$$

for $\alpha = 0.5$

$$\begin{gathered}\varphi_1 = \sqrt{2\pi}\xi S(\xi) + \cos\xi^2, \quad \varphi_2 = \xi, \\ \varphi_3 = 0.5[\sqrt{2\pi}\xi C(\xi) - \sin\xi^2], \quad \varphi_4 = 0.125\{2\pi\xi[S(\xi)]^2 + 2\pi\xi[C(\xi)]^2 + 2\sqrt{2\pi}[S(\xi)\cos\xi^2 - C(\xi)\sin\xi^2]\}.\end{gathered} \tag{2.8}$$

We note that the solution of (2.7) and the first three solutions (2.8) equal the derivatives with respect to $\xi$ of (2.6) and (2.7), respectively, to within an arbitrary factor.

The particular solutions (2.6)-(2.8) are chosen so that for $\xi = 0$ the values of these functions and their third-order derivatives vanish and equal unity, except for the values indicated below:

$$\varphi_1(0) = \varphi_2'(0) = \varphi_3''(0) = \varphi_4'''(0) = 1.$$

This choice of functions simplifies the satisfaction of the boundary conditions at $\xi = 0$ by the solutions.

In studying the bending of infinite and semiinfinite beams, it is necessary to use solutions that approach zero for $\xi \to \infty$. These solutions can be obtained by using a linear combination of the fundamental system of solutions written. For example, for $\alpha = 1.5$, these solutions will be

$$\begin{gathered}\varphi_5 = \varphi_1 - 0.75\sqrt{2\pi}\varphi_2 + 3\sqrt{2\pi}\varphi_4 = \\ = -\sqrt{2\pi}\xi^3[C(\xi) - 0.5] + 1.5\sqrt{2\pi}\xi[S(\xi) - 0.5] + \xi^2\sin\xi^2 + \\ + \cos\xi^2, \quad \varphi_6 = 6\varphi_3 - 0.75\sqrt{2\pi}\varphi_2 - 3\sqrt{2\pi}\varphi_4 = \\ = \sqrt{2\pi}\xi^3[S(\xi) - 0.5] + 1.5\sqrt{2\pi}\xi[C(\xi) - 0.5] + \xi^2\cos\xi^2 - \sin\xi^2.\end{gathered} \tag{2.9}$$

The results obtained permit, without difficulty, reproducing known self-similar solutions. For example, the solution of the problem of bending of an infinite beam under the action of a force (2.3), examined in [3], can be written in the form

$$w = \frac{2}{3\sqrt{2\pi}} \frac{P_* a^{\frac{3}{4}}}{EI} t^{\frac{3}{2}} (\varphi_5 - \varphi_6). \tag{2.10}$$

In obtaining (2.10), we used the general solution of Eq. (2.2) for $\alpha = 1.5$, approaching zero for $\xi \to \infty$:

$$\varphi = C_5 \varphi_5 + C_6 \varphi_6,$$

where $C_5$ and $C_6$ are arbitrary constants, determined from the boundary conditions at $\xi = 0$.

3. The proposed method does not permit obtaining solutions for beams with finite length. However, it is possible to consider problems in which the supports are moving. The law for the displacement of the supports must be such that it does not introduce additional quantities with independent dimensionalities. The following law of motion satisfies this condition:

$$x = b2\sqrt[4]{a}\sqrt{t}, \tag{3.1}$$

where b is a dimensionless constant, characterizing the rate at which the supports are displaced.

Condition (3.1) determines the boundary conditions for Eq. (2.2) for the value $\xi = b$. In other words, in the region of the variable $\xi$, we can consider the bending of some fictitious beam whose deformation is described by the more complicated equation (2.2). In sections $\xi = b_1, b_2, b_3, \ldots$ of this beam, there are supports for applied forces and moments. In addition, a distribution of the loads can be given. The methods for finding the solutions for a fictitious beam do not essentially differ from the methods of calculating static bending of beams [11].

Transforming again to the variables x and t, we obtain a Bernoulli–Euler beam with moving supports and moving loads, running from the point $x = 0$ in the self-similar regime, determined by equations of the form (3.1).

Many works are concerned with questions of bending of beams under the action of moving loads. Numerical methods can be used in obtaining solutions [12]. For infinite beams (taking into account the elastic foundation and damping), analytic solutions are sought for action of constant loads, running along beams with constant velocity. In this case, it is possible to construct an asymptotically exact solution for long times from the moment that the loads begin to move [13, 14]. However, the initial transient process is difficult to study and requires considerable computational work.

The application of self-similar solutions permits constructing a simple exact solution of the problem of a running load from the very beginning of its motion. In this case, the load can vary with time according to a power law.

Let us examine the bending of an infinite beam under the action of a concentrated force, moving along the beam from the origin of the coordinate $x = 0$ according to the law (3.1). This problem can be split up into two problems. In these problems, two forces identical in magnitude move away from the origin of coordinates in opposite directions, and in addition, in the first problem (symmetrical problem) the forces are directed in the same direction, while in the second problem in opposite directions (antisymmetrical problem). It is easy to formulate the boundary conditions at $x = 0$ for each problem. The antisymmetrical problem corresponds to free resting on a rigid support. In the symmetrical problem, limitations are imposed on the angle of rotation, which must remain equal to 0.

Let us examine the behavior of this solution for the antisymmetric problem. We assume that the force varies in time according to (2.3). We choose null initial conditions. The deflection is sought in the form (2.1), where $\alpha = 1.5$, while the coefficient (A) is determined by (2.5). Substituting (2.1) into the boundary conditions of the problem for $x = 0$ and $x \to \infty$, we obtain

$$\varphi = \varphi'' = 0 \ (\xi = 0), \ \varphi \to 0 \ (\xi \to \infty). \tag{3.2}$$

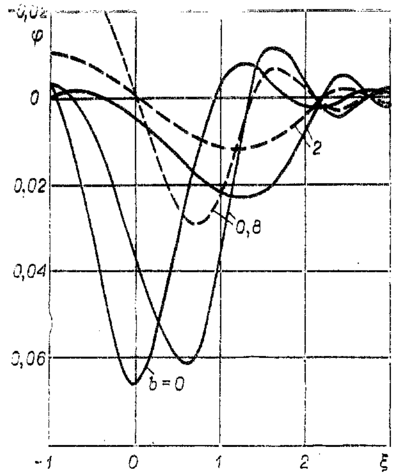


Fig. 1

In the section beneath the force, the deflection and its first and second derivatives are continuous, i.e., $\varphi$, $\varphi'$, and $\varphi''$ are continuous, while the third derivative is discontinuous and, taking into account (2.4) and (2.5), satisfies

$$\varphi'''(b+0) - \varphi'''(b-0) = 1.$$

Let use use the particular solutions (2.6) and (2.9) for $\alpha = 1.5$. The general solution is written in the form

$$\varphi = C_2\varphi_2 + C_4\varphi_4\ (\xi \leqslant b),\ \varphi = C_5\varphi_5 + C_6\varphi_6\ (\xi \geqslant b). \tag{3.3}$$

The first and second solution (3.3) satisfy boundary conditions (3.2), respectively, at $\xi = 0$ and $\xi \to \infty$. The arbitrary constants $C_2$, $C_4$, $C_5$, and $C_6$ are found from the above formulated conditions for the joining of the deflection and its derivatives at the points of application of the force ($\xi = b$).

The calculations were carried out for b = 0.8 and 2. In doing the calculations, we used the tables in [15]. Figure 1 shows the graphs of the function $\varphi(\xi)$ for the antisymmetrical problem (dashed curves) and the total solution of the antisymmetrical and symmetrical problems (continuous curves). The total solution corresponds to the action of an infinite unsupported beam of the doubled force $2P_*$ moving from the section x = 0 in one direction. For comparison according to formula (2.10), we also compute the curve $\varphi$ for an infinite beam with action of a stationary force $2P_*$ (b = 0).

The bending moment equals

$$M = EI\frac{\partial^2 w}{\partial x^2} = 0.25EIAt^{\alpha-1}a^{-\frac{1}{2}}\varphi'' = 2P_*\sqrt[4]{a}\ \sqrt{t}\,\varphi''.$$

The curves $\varphi''$ are constructed in Fig. 2. The curves are labeled in the same manner as in Fig. 1. The curves shown in Figs. 1 and 2 indicate the fact that as the rate at which the forces are displaced in the self-similar regime increases, the deflection and bending moments decrease. The motion of the force along the beam occurs more rapidly than the increase in the deflection, and in addition, for velocities corresponding to b > 2, the deflection under the force is close to zero. The bending moment decreases rapidly under the force with increasing b. However, the bending moment in the precursor (wave, traveling in front of the force) decreases more slowly.

Comparison of the solutions of the symmetrical and antisymmetrical problems shows that for b > 2 the nature of the boundary conditions on the support at x = 0 has little effect on the deformation state of the beam in the region of application of the force.

As the rate of displacement of the force increases (b > 2) computational difficulties arise, related to the necessity of calculating small differences in functions (2.9) and their derivatives when using these functions in (3.3). In order to simplfy the calculations, let us transform Eq. (2.2). We introduce the new variable

$$\xi_* = \xi - b.$$

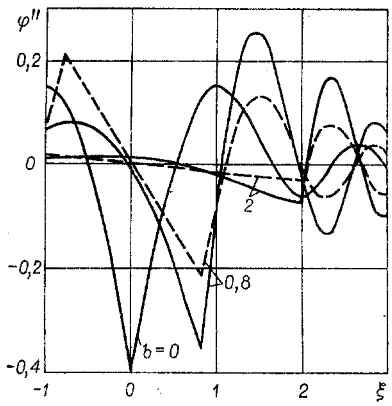


Fig. 2

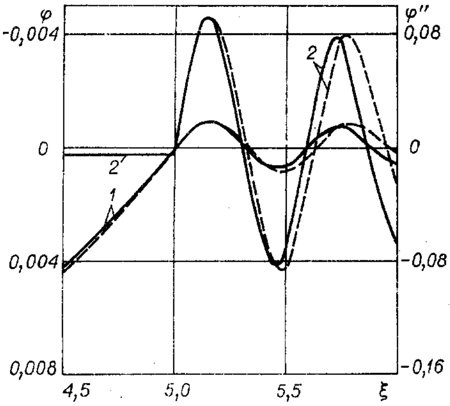


Fig. 3

For the new variable, the equation takes the form

$$\varphi^{IV} + 4\varphi''(\xi_* + b)^2 + (12 - 16\alpha)\varphi'(\xi_* + b) + 16\alpha(\alpha - 1)\varphi = 0. \quad (3.4)$$

Here, the derivatives are taken with respect to the variable $\xi_*$.

If $|\xi_*| << b$, then $\xi_* + b \approx b$ and Eq. (3.4) transforms into a linear differential equation with constant coefficients, whose solution is elementary. For $\alpha = 1.5$, Eq. (3.4) has the form

$$\varphi^{IV} + 4b^2\varphi'' - 12b\varphi' + 12\varphi = 0. \quad (3.5)$$

In searching for the deformation of the beam in the vicinity of a moving force instead of $\varphi_5$ and $\varphi_6$ in (3.3) we used two corresponding particular solutions of Eqs. (3.5). The calculations show that the use of the approximate Eq. (3.5) with b = 2 still gives a significant error for the deflection and bending moment in the region of application of the force, but as the magnitude of b increases, the error rapidly decreases.

Figure 3 illustrates φ (curve 1) and φ" (curve 2) in the region of application of a moving force $P_*$ with b = 5. The solid line shows the exact solution and the dashed line corresponds to the use of the approximate Eq. (3.5). The graphs indicate the good agreement of the solutions.

In conclusion, we note that the use of concepts of self-similar regimes of motion of moving loads may turn out to be useful in studying other problems in mathematical physics, which admit self-similar solutions and the possibility of introducing moving sources of perturbations.

## LITERATURE CITED


1. L. I. Sedov, Similarity and Dimensional Methods in Mechanics [in Russian], Nauka, Moscow (1972).
2. P. Duve, D. Clark, and Kh. Bonenblyust, "Behavior of long beams under impact loads," [Russian translation], Collection of Translations in Mechanics, No. 3 (1950).
3. A. I. Lur'e, Operational Calculation and Its Application in Mechanical Problems [in Russian], Gostekhizdat, Moscow (1951).
4. Pol and Fu, "Semiinfinite beam with a velocity discontinuity given at the end face," Prikl. Mekh., No. 1 (1967).
5. C. C. Fu, "Closed-form solutions of an infinite beam under impact loading," Int. J. Solids Struct., 3 (1967).
6. D. Muchichescu, "Sur la propagation des ondes transversales dans une barre élastique infinie," Rev. Roumaine Math. Pures. Appl., XV, No. 2 (1970).
7. A. I. Tseitlin, "Oscillations of semiinifinte rods and cylindrical shells," in: Dynamics of Structures [in Russian], Izd. V. A. Kucherenko TsNII, Moscow (1971).
8. L. I. Slepyan, Nonstationary Elastic Waves [in Russian], Sudostroenie, Leningrad (1972).
9. Yu. P. Samarin, "Application of self-similar solutions to the problem of oscillations of an elastic membrane expanding from a point," in: Mechanics [in Russian], Izd. V. V. Kuibyshev Polytechnical Institute, Kuibyshev (1967).
10. I. M. Ryzhik and I. S. Gradshtein, Table of Integrals, Series, and Products, Academic Press (1966).

11. Handbook for Shop Structural Mechanics [in Russian], Vol. 1, Sudpromgiz, Leningrad (1958).
12. A. P. Filippov and S. S. Kokhmanyuk, Dynamic Action of Moving Loads on Rods [in Russian], Naukova Dumka, Kiev (1967).
13. W. Stadler and R. E. Shreeves, "The transient and steady-state response of the infinite Bernoulli—Euler beams with damping and elastic foundation," Q. J. Mech. Appl. Math., 23, Pt. 2 (1970).
14. F. Ladislaw, Vibration of Solids and Structure under Moving Loads, Czech. Acad. Sci., Prague (1972).
15 Tables of Fresnel Integrals [in Russian], Izv. Akad. Nauk SSSR, Moscow (1953).